Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 6, pp. 1064–1072, 2001

SOLUTION OF THE PLANE HERTZ PROBLEM

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UDC 539.3

The paper considers the problem of one-sided frictionless compression of plane elastic bodies that are initially in contact with each other at a point. The first terms of an asymptotic solution of the problem are constructed by the method of joined asymptotic expansions. Determination of the approach of the bodies as a function of the pressing force reduces to calculating so-called of local compliance. The problems of contact of an elastic ring and elastic circular disks with punches and an elastic disk compressed between two elastic strips are considered. An asymptotic model for the quasistatic collision of plane elastic bodies is proposed.

The problems of one-sided contact of elastic bodies have been studied within the framework of the theory of variational inequalities (see, e.g., [1–4]). A finite-element algorithm for solving problem (1.8) was developed in [1]. In the present paper, an approximate solution of the problem is constructed by the method of joined asymptotic expansions [5–7]. This method was used in [8] to solve the problem of compression of two circular disks and in [9] to analyze the indentation of an elastic disk into a rigid corner with allowance for friction. The problem considered below was solved in [10] to study the strength of valves for channels of small flow section.

1. Formulation of the Problem. We consider two elastic bodies Ω^1 and Ω^2 pressed into one another (Fig. 1). We denote the displacement vector of the body Ω^r ($r = 1,\ 2$) by $\boldsymbol{u}^r = (u_1^r, u_2^r)$. It is assumed that the body Ω^1 is fixed along the part of the boundary Γ_u, i.e.,

$$\boldsymbol{u}^1(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} = (x_1, x_2) \in \Gamma_u. \tag{1.1}$$

In view of the symmetry of the problem, on the segment Γ_0 of the boundary of the body Ω^2, we specify the conditions of two-sided contact:

$$u_2^2(\boldsymbol{x}) = 0, \quad \tau_{12}^2(\boldsymbol{u}^2; \boldsymbol{x}) = 0, \qquad \boldsymbol{x} \in \Gamma_0. \tag{1.2}$$

On the segment Γ_τ^2, surface loads are specified:

$$\sum_{j=1}^{2} \tau_{ij}^2(\boldsymbol{u}^2; \boldsymbol{x}) n_j^2(\boldsymbol{x}) = q_i(\boldsymbol{x}) \quad (i = 1, 2), \qquad \boldsymbol{x} \in \Gamma_\tau^2. \tag{1.3}$$

For simplicity, we assume that the segment Γ_τ^1 of the boundary of the body Ω^1 is free of stress:

$$\sum_{j=1}^{2} \tau_{ij}^1(\boldsymbol{u}^1; \boldsymbol{x}) n_j^1(\boldsymbol{x}) = 0 \quad (i = 1, 2), \qquad \boldsymbol{x} \in \Gamma_\tau^1. \tag{1.4}$$

Here $\tau_{ij}^r(\boldsymbol{u}^r)$ are the stress-tensor components corresponding to the displacement vector $\boldsymbol{u}^r$ and e_{ij}^r are the strain-tensor components of the body Ω^r:

$$\tau_{ij}^r = 2\mu_r e_{ij}^r + \delta_{ij}\mu_r \frac{3 - æ_r}{æ_r - 1}(e_{11}^r + e_{22}^r), \qquad e_{ij}^r = \frac{1}{2}\Big(\frac{\partial u_i^r}{\partial x_j} + \frac{\partial u_j^r}{\partial x_i}\Big),$$

where δ_{ij} is the Kronecker symbol, μ_r is the shear modulus, $æ_r = 3 - 4\nu_r$ (for plane strain), and ν_r is Poisson's ratio.

Makarov State Marine Academy, St. Petersburg 199106. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 6, pp. 166–176, November–December, 2001. Original article submitted October 3, 2000; revision submitted April 10, 2001.

0021-8944/01/4206-1064 $25.00

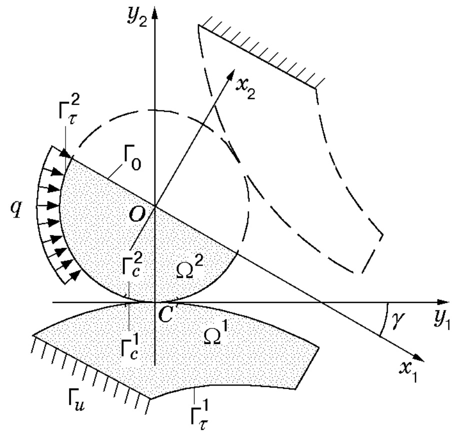

Fig. 1

In the undeformed state, the elastic bodies Ω^1 and Ω^2 contact each other at the point C, which is the origin of the local coordinates y_1 and y_2. Under the external load, the body Ω^2 is pressed into Ω^1. For simplicity, we assume that there is no friction between the surfaces in contact is absent. The segments of the boundaries Γ_c^1 and Γ_c^2 that can come into contact during the deformation are smooth curves defined by the equations

$$y_2 = f_r(y_1), \qquad y_1 \in (-l_1, l_2) \qquad (r = 1, 2), \tag{1.5}$$

where $(-l_1, l_2)$ is the interval containing the projection of the possible contact zone. We determine the distance between the points of the bodies in contact in the initial state (before deformation)

$$\Delta(y_1) = f_2(y_1) - f_1(y_1). \tag{1.6}$$

To obtain the boundary conditions of one-sided contact on Γ_c^1 and Γ_c^2, we consider the variational formulation of the problem [1]. We introduce the space of possible displacement fields with finite energy $V = \{(\boldsymbol{v}^1, \boldsymbol{v}^2)\colon \boldsymbol{v}^1 = 0$ on Γ_u, $v_2^2 = 0$ on $\Gamma_0\}$ (see [1]) and define the potential energy functional

$$L(\boldsymbol{v}^1, \boldsymbol{v}^2) = \sum_{r=1}^{2} \frac{1}{2} \int\limits_{\Omega^r} \sum_{i,j=1}^{2} \tau_{ij}^r(\boldsymbol{v}^r; \boldsymbol{x}) e_{ij}(\boldsymbol{v}^r; \boldsymbol{x})\, d\boldsymbol{x} - \int\limits_{\Gamma_\tau^2} \sum_{i=1}^{2} q_i(\boldsymbol{x}) v_i^2(\boldsymbol{x})\, ds_x.$$

According to [1], the contact problem of compression of two bodies Ω^1 and Ω^2 which are initially in contact at a point reduces to minimization of the functional $L(\boldsymbol{v}^1, \boldsymbol{v}^2)$ on the set of admissible displacements

$$K_\Delta = \Big\{(\boldsymbol{v}^1, \boldsymbol{v}^2) \in V\colon \hat{v}_2^1(y_1, f_1(y_1)) - \hat{v}_2^2(y_1, f_2(y_1)) \leqslant \Delta(y_1),\ y_1 \in (-l_1, l_2)\Big\}.$$

Here $\hat{v}_2^r = -v_1^r \sin\gamma + v_2^r \cos\gamma$ is the projection of the vector $\boldsymbol{v}^r$ onto the Oy_2 axis and γ is the angle between the x_1 and y_1 axes.

The following statement is valid (see [1, Theorem 2.6]). Let Γ_0 be composed of segments parallel to the Ox_1 axis. If $\cos(x_1, y_2) < 0$ and

$$Q_1 = \int\limits_{\Gamma_\tau^2} q_1(\boldsymbol{x})\, ds_x > 0, \tag{1.7}$$

there exists a unique solution $(\boldsymbol{u}^1, \boldsymbol{u}^2) \in K_\Delta$ of the problem

$$L(\boldsymbol{u}^1, \boldsymbol{u}^2) \leqslant L(\boldsymbol{v}^1, \boldsymbol{v}^2) \qquad \forall (\boldsymbol{v}^1, \boldsymbol{v}^2) \in K_\Delta. \tag{1.8}$$

Furthermore, at points of the curves Γ_c^1 and Γ_c^2 with equal abscissa $y_1 \in (-l_1, l_2)$ the following relations hold:

$$\begin{gathered} \hat{u}_2^1(y_1, f_1) - \hat{u}_2^2(y_1, f_2) \leqslant \Delta(y_1), \qquad \hat{T}_2^1(y_1, f_1)/\cos\alpha_1 = -\hat{T}_2^2(y_1, f_2)/\cos\alpha_2 \leqslant 0, \\ [\hat{u}_2^1(y_1, f_1) - \hat{u}_2^2(y_1, f_2) - \Delta(y_1)]\hat{T}_2^1(y_1, f_1) = 0; \end{gathered} \tag{1.9}$$

$$\hat{T}_1^1(y_1, f_1) = \hat{T}_1^2(y_1, f_2) = 0. \tag{1.10}$$

Here $\hat{T}_i^r$ is the projection of the stress vector at a site with normal $\boldsymbol{n}^r$ onto the Oy_i axis and α_r is the angle between the Oy_1 axis and the tangent to Γ_c^r. Satisfaction of inequality (1.7) means that the body Ω^2 is pressed into Ω^1.

To use the asymptotic method, we introduce the small parameter ε

$$q_i = \varepsilon q_i^* \tag{1.11}$$

and assume that the body Ω^2 is subjected to a force system with the resultant

$$Q_1 = \varepsilon Q_1^*, \qquad Q_1^* = \int\limits_{\Gamma_\tau^2} q_1^*(\boldsymbol{x})\, ds_x$$

proportional to the parameter ε and directed along the Ox_1 axis.

Following [11], we construct the leading terms of the asymptotic solution of the problem (1.8), (1.11) as $\varepsilon \to 0$.

2. External Asymptotic Representation. We denote by $\boldsymbol{G}^1(\boldsymbol{x})$ the solution of the problem of the body Ω^1 loaded at the point C by a unit force directed along the inward normal to the boundary of Ω^1. The vector $\boldsymbol{G}^1(\boldsymbol{x})$ must satisfy the Lamé equations in Ω^1, the boundary condition (1.1) on the segment Γ_u, and the condition that the segments Γ_τ^1 and $\Gamma_c^1 \setminus C$ are stress-free.

The projections of the vector $\boldsymbol{G}^1(\boldsymbol{x})$ onto the axes of the local coordinate system Cy_1y_2 are given by

$$\hat{G}_1^1(\boldsymbol{y}) = G_1^1(\boldsymbol{x})\cos\gamma + G_2^1(\boldsymbol{x})\sin\gamma, \qquad \hat{G}_2^1(\boldsymbol{y}) = -G_1^1(\boldsymbol{x})\sin\gamma + G_2^1(\boldsymbol{x})\cos\gamma, \tag{2.1}$$

where $x_1 = y_1\cos\gamma - y_2\sin\gamma$ and $x_2 = y_1\sin\gamma + y_2\cos\gamma$.

As $|\boldsymbol{y}| = (y_1^2 + y_2^2)^{1/2} \to 0$, we obtain the asymptotic formulas

$$\hat{G}_i^1(\boldsymbol{y}) = S_i^1(\boldsymbol{y}/R_1) + n_1 A_i^1 + O(|\boldsymbol{y}|) \qquad (i = 1, 2). \tag{2.2}$$

Here R_1 is the radius of curvature of the contour Γ_c^1 at the point C, $\boldsymbol{S}^r(\boldsymbol{y}/R_1)$ is a solution of the Flamant problem (see, e.g., [12, § 90]) of the elastic half-plane $y_2 \leqslant 0$ (for $r = 1$) loaded by a unit point force in opposition to the Cy_2 axis, and

$$\begin{aligned} 4\pi\mu_r S_1^r(\boldsymbol{\zeta}) &= -2\zeta_1\zeta_2/|\boldsymbol{\zeta}|^2 + (æ_r - 1)\arctan(\zeta_1/\zeta_2), \\ 4\pi\mu_r S_2^r(\boldsymbol{\zeta}) &= (æ_r + 1)\ln|\boldsymbol{\zeta}| - 2\zeta_2^2/|\boldsymbol{\zeta}|^2, \qquad n_r = (æ_r + 1)/(4\pi\mu_r). \end{aligned} \tag{2.3}$$

Here $\boldsymbol{\zeta} = (\zeta_1, \zeta_2)$ are dimensionless coordinates.

At a distance from the contact site, the displacement field $\boldsymbol{u}^1(\boldsymbol{x})$ of the body Ω^1 is written as

$$\boldsymbol{v}^1(\boldsymbol{x}) = P\boldsymbol{G}^1(\boldsymbol{x}), \tag{2.4}$$

where P is the contact force.

We consider the body Ω^2. To describe its displacement field $\boldsymbol{u}^2(\boldsymbol{x})$ at a distance from the contact site, we replace the action of the body Ω^1 on Ω^2 by the point force P directed along the Cy_2 axis. As a result, we arrive at the problem of equilibrium of the body Ω^2 which lies on the smooth support Γ_0 and is loaded by a self-balanced force system. The solution of this problem is determined with accuracy up to translational displacement.

We introduce a normalized solution of this problem $\boldsymbol{v}^{20}(\boldsymbol{x})$ which is uniquely determined by the condition $v_1^{20}(O) = 0$ having simple mechanical meaning: slip is absent at the point O. The vector $\boldsymbol{v}^{20}(\boldsymbol{x})$ must satisfy the homogeneous Lamé equations in the region Ω^2, the boundary conditions of two-sided contact on Γ_0 (1.2), the force boundary condition on Γ_τ^2 (1.3), and the asymptotic condition at the limit $\boldsymbol{x} \to C$ [cf. Eq. (2.2)]:

$$\hat{v}_i^{20}(\boldsymbol{y}) = -P[S_i^2(\boldsymbol{y}/R_2) + n_2 A_i^2 + O(|\boldsymbol{y}|)] \qquad (i = 1, 2). \tag{2.5}$$

Here R_2 is the radius of curvature of Γ_c^2 at the point C and $\hat{v}_i^{20}(\boldsymbol{y})$ is the projection of the vector $\boldsymbol{v}^{20}(\boldsymbol{x})$ onto the Cy_i axis [see Eqs. (2.1)].

The equation of static equilibrium of the body Ω^2 yields

$$P = \varepsilon P^*, \qquad P^* = (\sin\gamma)^{-1} Q_1^*. \tag{2.6}$$

Finally, the external asymptotic representation for the displacement vector of the elastic body Ω^2 is given by

$$\boldsymbol{v}^2(\boldsymbol{x}) = \boldsymbol{v}^{20}(\boldsymbol{x}) + \alpha \boldsymbol{e}_1, \tag{2.7}$$

where the constant α is a small displacement of the center of the body Ω^2 along the Ox_1 axis. Let

$$\alpha = \varepsilon \alpha^*. \tag{2.8}$$

The quantity α is determined by local strains in the contact zone.

3. Problem of One-Sided Contact for the Boundary Layer. In the neighborhood of the initial contact point, the boundaries Γ_c^1 and Γ_c^2 are approximated by the parabolas

$$f_r(y_1) = (-1)^r (2R_r)^{-1} y_1^2 + O(y_1^3) \qquad (r = 1, 2). \tag{3.1}$$

We introduce the "extended" coordinates

$$\boldsymbol{\eta} = (\eta_1, \eta_2), \qquad \eta_i = \varepsilon^{-1/2} y_i. \tag{3.2}$$

The degree of extension in (3.2) is chosen so that the dimensions of the contact site do not depend on the parameter ε in the coordinates η_1 and η_2 (see existing solutions of the problem of a parabolic punch pressed into an elastic half-plane [12, 13]).

Accordingly, we express the magnitude of the clearance between the surfaces of the bodies Ω^1 and Ω^2 in the undeformed state [see Eq. (1.6)]:

$$\Delta(\varepsilon^{1/2}\eta_1) = \varepsilon[(2R_1)^{-1} + (2R_2)^{-1}]\eta_1^2 + O(\varepsilon^{3/2}\eta_1^3). \tag{3.3}$$

At the same time, using (1.5) and (3.1), we obtain the asymptotic representation of the boundaries Γ_c^1 and Γ_c^2 in the coordinates (3.2):

$$\eta_2 = (-1)^r \varepsilon^{1/2} (2R_r)^{-1} \eta_1^2 \qquad (r = 1, 2). \tag{3.4}$$

Because in conversion to the coordinates (3.2), the ends of the arcs Γ_c^1 and Γ_c^2 are shifted from the point C by the distances $\varepsilon^{-1/2} l_1$ and $\varepsilon^{-1/2} l_2$, respectively, the internal asymptotic representation $\boldsymbol{w}^r(\boldsymbol{\eta})$ of the displacement field $\boldsymbol{u}^r(\boldsymbol{x})$ of the elastic body Ω^r is formulated in a semi-infinite region with the parabolic boundary (3.4).

According to the method of joined asymptotic expansions, formulas (2.2) and (2.5) [with allowance for (2.4) and (2.7)] define the behavior of the vector functions $\boldsymbol{w}^1(\boldsymbol{\eta})$ and $\boldsymbol{w}^2(\boldsymbol{\eta})$ at infinity. Letting $|\boldsymbol{\eta}| \to \infty$ and ignoring terms $O(\varepsilon^{1/2}|\boldsymbol{\eta}|)$ in (2.2) and (2.5), we obtain

$$\boldsymbol{w}^1(\boldsymbol{\eta}) = \varepsilon P^* \Big[\boldsymbol{S}^1(\varepsilon^{1/2}\boldsymbol{\eta}/R_1) + n_1 \boldsymbol{A}^1\Big] + O(|\boldsymbol{\eta}|^{-1}); \tag{3.5}$$

$$\boldsymbol{w}^2(\boldsymbol{\eta}) = -\varepsilon P^* \Big[\boldsymbol{S}^2(\varepsilon^{1/2}\boldsymbol{\eta}/R_2) + n_2 \boldsymbol{A}^2\Big] + \varepsilon\alpha^*(\cos\gamma, -\sin\gamma) + O(|\boldsymbol{\eta}|^{-1}). \tag{3.6}$$

In Eqs. (3.5) and (3.6), the normalizing relations (2.6) and (2.8) are used.

We obtain boundary conditions for the vectors $\boldsymbol{w}^1(\boldsymbol{\eta})$ and $\boldsymbol{w}^2(\boldsymbol{\eta})$. To this end, we substitute relations (3.1) into the boundary conditions of one-sided frictionless contact (1.9) and (1.10) and make the change of variables inverse to (3.2). As a result, with allowance for (3.4), the nonpenetration condition for the bodies in contact [see the first inequality in (1.9)] becomes

$$w_2^1(\eta_1, \varepsilon^{1/2}\varphi_1(\eta_1)) - w_2^2(\eta_1, \varepsilon^{1/2}\varphi_2(\eta_1)) \leqslant \varepsilon\Delta^*(\eta_1). \tag{3.7}$$

Here $\varphi_r(\eta_1) = (-1)^r(2R_r)^{-1}\eta_1^2$ $(r = 1, 2)$ and $\Delta^*(\eta_1) = [(2R_1)^{-1} + (2R_2)^{-1}]\eta_1^2$. Similarly to (3.7), one can transform the remaining relations in (1.9) and (1.10).

4. Internal Asymptotic Representation. In the local-strain region, the solution can be written as

$$\boldsymbol{w}^1(\boldsymbol{\eta}) = \varepsilon \boldsymbol{W}^1(\boldsymbol{\eta}) + \varepsilon P^* n_1(\boldsymbol{A}^1 + \hat{\boldsymbol{e}}_2 \ln\sqrt{\varepsilon}); \tag{4.1}$$

$$\boldsymbol{w}^2(\boldsymbol{\eta}) = \varepsilon \boldsymbol{W}^2(\boldsymbol{\eta}) - \varepsilon P^* n_2(\boldsymbol{A}^2 + \hat{\boldsymbol{e}}_2 \ln\sqrt{\varepsilon}) + \varepsilon\alpha^*(\cos\gamma, -\sin\gamma), \tag{4.2}$$

where $\hat{\boldsymbol{e}}_2$ is the unit vector of the $C\eta_2$ axis.

Substitution of (4.1) and (4.2) into (3.7) yields

$$
\begin{gathered}
W_2^1(\eta_1, \varepsilon^{1/2}\varphi_1(\eta_1)) - W_2^2(\eta_1, \varepsilon^{1/2}\varphi_2(\eta_1)) \leqslant \Delta_\varepsilon^*(\eta_1), \\
\Delta_\varepsilon^*(\eta_1) = \Delta^*(\eta_1) - P^* \sum_{r=1}^{2} n_r (A_2^r + \ln\sqrt{\varepsilon}) - \alpha^* \sin\gamma.
\end{gathered}
\tag{4.3}
$$

Finally, we obtain the boundary condition for the leading terms of the asymptotic representation of the vector functions $\boldsymbol{W}^1(\boldsymbol{\eta})$ and $\boldsymbol{W}^2(\boldsymbol{\eta})$ by passing to the limit on the left side of inequality (4.3) as $\varepsilon \to 0$ and "straightening" the boundaries (3.4).

Thus, the vectors $\boldsymbol{W}^1(\boldsymbol{\eta})$ and $\boldsymbol{W}^2(\boldsymbol{\eta})$ must satisfy the Lamé equations in the half-planes $\eta_2 < 0$ and $\eta_2 > 0$, respectively, and the following conditions at infinity ($r = 1, 2$):

$$
\boldsymbol{W}^r(\boldsymbol{\eta}) = (-1)^{r+1} P^* \boldsymbol{S}^r(\boldsymbol{\eta}/R_r) + O(|\boldsymbol{\eta}|^{-1}), \qquad |\boldsymbol{\eta}| \to \infty.
\tag{4.4}
$$

Moreover, the relations

$$
\begin{gathered}
W_2^1(\eta_1, 0) - W_2^2(\eta_1, 0) \leqslant \Delta_\varepsilon^*(\eta_1), \qquad \tau_{22}^1(\eta_1, 0) = \tau_{22}^2(\eta_1, 0) \leqslant 0, \\
\Big[W_2^1(\eta_1, 0) - W_2^2(\eta_1, 0) - \Delta_\varepsilon^*(\eta_1)\Big] \tau_{22}^1(\eta_1, 0) = 0, \qquad \tau_{12}^1(\eta_1, 0) = \tau_{12}^2(\eta_1, 0) = 0
\end{gathered}
\tag{4.5}
$$

must be satisfied at the boundary $\eta_2 = 0$.

One can easily obtain the exact solution of this problem using the results of [12, 13]. We denote the half-width of the unknown contact zone [in the coordinates (3.2)] by h_*. We express the vector function $\boldsymbol{W}^r(\boldsymbol{\eta})$ in terms of the integral

$$
\boldsymbol{W}^r(\boldsymbol{\eta}) = (-1)^{r+1} \int\limits_{-h_*}^{h_*} \boldsymbol{S}\Big(\frac{\eta_1 - \xi}{R_r}, \frac{\eta_2}{R_r}\Big) p^*(\xi)\, d\xi
\tag{4.6}
$$

with the density

$$
p^*(\eta_1) = (2P^*/(\pi h_*))\sqrt{1 - \eta_1^2/h_*^2}.
\tag{4.7}
$$

The vector (4.6) satisfies the asymptotic condition (4.4). At the same time, the equality

$$
W_2^r(\eta_1, 0) = (-1)^r P^* n_r \Big(\frac{\eta_1^2}{h_*^2} - \ln\frac{2R_r}{h_*} - \frac{1}{2}\Big) \quad [\eta_1 \in (-h_*, h_*)]
\tag{4.8}
$$

holds on the boundary of the half-plane. Substituting the expression for $\Delta_\varepsilon^*(\eta_1)$ from (4.3) and the boundary values (4.8) into the displacement-compatibility equation $W_2^1(\eta_1, 0) - W_2^2(\eta_1, 0) = \Delta_\varepsilon^*(\eta_1)$, where $\eta_1 \in (-h_*, h_*)$, after simple manipulations we obtain the system

$$
h_*^2 = (2R_1R_2/(R_1 + R_2))P^*(n_1 + n_2);
\tag{4.9}
$$

$$
P^* \sum_{r=1}^{2} n_r \Big(\ln\frac{2R_r}{\sqrt{\varepsilon} h_*} + \frac{1}{2} - A_2^r\Big) = \alpha^* \sin\gamma,
\tag{4.10}
$$

where n_r is the elastic constant given in (2.3) and $2h_*$ is the dimension of the contact zone in the coordinates (3.2). Reverting to the real scale, from (3.2) we obtain

$$
h = \sqrt{\varepsilon} h_*.
\tag{4.11}
$$

Thus, we have obtained formulas (4.7), (4.9), and (4.10) by constructing the leading terms of the asymptotic solution of the initial contact problem. As one would expect, Eqs. (4.7) and (4.9) coincide with the Hertz solution. Equation (4.10) relating the force P^* to the displacement α^* is a new result.

5. Asymptotic Modeling of Compressed Elastic Bodies. Sample Problems. The contact pressure and the characteristic contact parameters h and α are determined from the following equations [see (2.6), (2.8), (4.7), and (4.9)–(4.11)]:

$$
p(y_1) = (2P/(\pi h))\sqrt{1 - y_1^2/h^2}, \qquad h^2 = (2R_1R_2/(R_1 + R_2))(n_1 + n_2)P;
\tag{5.1}
$$

$$
P \sum_{r=1}^{2} n_r \Big(\ln\frac{2R_r}{h} + \frac{1}{2} - A_2^r\Big) = \alpha \sin\gamma.
\tag{5.2}
$$

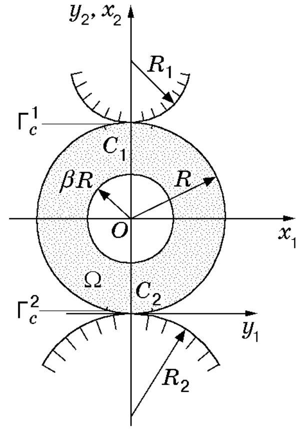

Fig. 2

TABLE 1

β	c
0.1	0.0926
0.2	0.4042
0.3	1.0638
0.4	2.4052
0.5	5.3122
0.6	12.4833
0.7	34.4229

Equation (5.2) contains the dimensionless constants A_2^1 and A_2^2 [coefficients in the asymptotic formulas (2.2) and (2.5)]. The quantity A_2^1 depends on the position of the point C and the shape and fixing conditions of the body Ω^1. The quantity A_2^2 depends on the distribution of the load acting on the body Ω^2. In particular cases, some of which are discussed below, these quantities can be expressed explicitly.

5.1. *Compression of an Elastic Ring by Curvilinear Punches.* We consider an elastic ring Ω with external radius R and internal radius βR ($0 \leqslant \beta < 1$) compressed by rigid punches (Fig. 2). For simplicity, the internal boundary of the ring Ω is assumed to be stress-free. We denote the mutual displacement of the punches by $2\delta_0$ and set $\delta_0 = \varepsilon\delta_0^*$, where ε is a small parameter. In this case, the condition of nonpenetration of the body Ω into the punch, for example, on the boundary Γ_c^2 [defined by the equation $y_2 = f_2(y_1)$] has the form $u_2(y_1, f_2) \geqslant \varepsilon\delta_0^* - [(2R_1)^{-1} + (2R_2)^{-1}]y_1^2$ for $|y_1| < l_2$.

We write the external asymptotic representation for the displacement field of the body Ω in the form $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}^0(\boldsymbol{x}) + \alpha\boldsymbol{e}_2$, where $\boldsymbol{v}^0(\boldsymbol{x})$ is a solution of the problem of compression of the elastic ring by forces P applied at the points C_1 and C_2. Using the explicit solution (see [14, Chapter 7, § 7.6]), we obtain expansion (2.5) for the components of the vector $\boldsymbol{v}^0(\boldsymbol{x})$ as $\boldsymbol{x} \to C_2$. In this case, we have $A_1 = 0$ and $A_2 = 1 - \ln 2 - c$, where

$$c = c_0 + \sum_{k=2,4,\dots} \frac{k}{k^2-1} c_k, \quad c_0 = \frac{\beta^2}{1-\beta^2}, \quad c_k = 2\beta^{2k-2}\,\frac{k^2(1-\beta^2)^2 + k(1-\beta^4) + 2\beta^2(1-\beta^{2k})}{(1-\beta^{2k})^2 - k^2\beta^{2k-2}(1-\beta^2)^2}.$$

Table 1 lists results obtained by these formulas.

As the internal asymptotic representation in the neighborhood of the point C_2, we use the sum (4.2) with $\gamma = \pi/2$. Simple calculations give

$$h_i^2 = 2RR_i(R+R_i)^{-1}nP \quad (i = 1, 2), \qquad n = (æ+1)/(4\pi\mu); \tag{5.3}$$

$$\alpha = \frac{1}{4} nP \ln\frac{R_2(R+R_1)}{R_1(R+R_2)}; \tag{5.4}$$

$$nP\Big(\ln(4R/h_1) + \ln(4R/h_2) - 1 + 2c\Big) = 2\delta_0. \tag{5.5}$$

Schwartz and Harper [8] considered the case of a circular disk ($\beta = 0$). Equations (5.3) and (5.5) (for $c = 0$) agree with results of [8]. The relative displacement of the center of the elastic disk (5.4) was not determined in [8]. For a circular disk, relation (5.5) was derived in [15] by another method [see § 5.6, formula (5.57)]. Mention should also be made of a paper [16], in which a two-term asymptotic representation was obtained for an elastic disk compressed by rectilinear punches. Relations (5.3) and (5.5) coincide with the leading terms of the asymptotic formulas (4.24) and (4.23), respectively, given in [16].

5.2. *Compression of Two Elastic Disks.* The contact pressure between the disks (Fig. 3) and the half-width of the contact zone h are given by formulas (5.1). For the half-width h_i of the contact of the disk with the punch, we have

$$h_i^2 = 2R_inP \qquad (i = 1, 2). \tag{5.6}$$

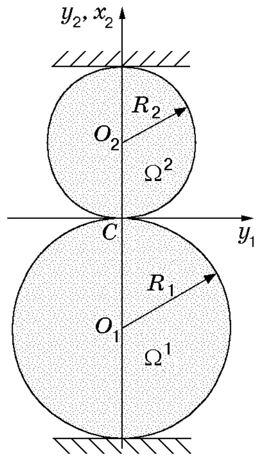

Fig. 3

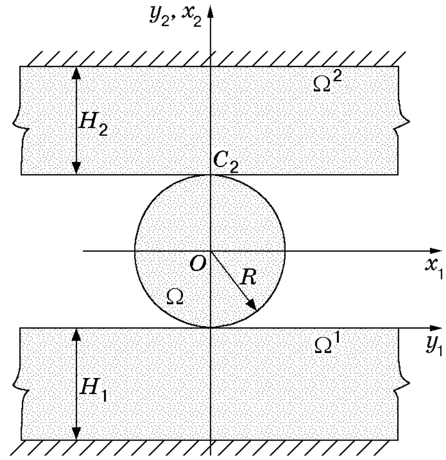

Fig. 4

The approach of the punches $2\delta_0$ is related to the compressing force P by the equation

$$P\sum_{i=1}^{2} n_i\Big(\ln\frac{4R_i}{h}+\ln\frac{4R_i}{h_i}-1\Big)=2\delta_0. \tag{5.7}$$

The decrease in the distance between the centers of the disks is given by

$$\alpha_1-\alpha_2=P\sum_{i=1}^{2} n_i\Big(\ln\frac{4R_i}{h}-\frac{1}{2}\Big), \tag{5.8}$$

where α_i is the displacement of the point O_i along the O_ix_2 axis.

To a first approximation, formulas (5.1) and (5.6)–(5.8) define the solution of the problem. An approximate relation between the mutual displacement of disks and the compressing force was first proposed in [17, Chapter 8]. Other solutions can be found in [18, § 54; 19]. The problem of two disks compressed by point forces was studied in [8, 20]. The solution of this problem is given by relations (5.1) and (5.8), which, in essence, coincide with the results of [8, 20]. It should be noted that formula (5.8) is exact in the asymptotic sense.

5.3. *An Elastic Disk Compressed Between Elastic Strips.* Let an elastic disk of radius R be compressed by elastic strips of widths H_1 and H_2 which are rigidly connected to punches (Fig. 4). As above, we denote the approach of the punches by $2\delta_0$. In this case, the external asymptotic representation for the displacement field of the first strip has the form $\boldsymbol{v}^1(\boldsymbol{x})=P\boldsymbol{G}^1(\boldsymbol{x})+\delta_0\boldsymbol{e}_2$. Here $\boldsymbol{G}^1(\boldsymbol{x})$ is a solution of the problem of the elastic body Ω^1 loaded at the point C_1 by a unit force in opposition to the Ox_2 axis and $\boldsymbol{G}^1(\boldsymbol{x})=0$ for $x_2=-(R+H_1)$.

In the asymptotic formula $G_j^1(y_1,y_2-R)=S_j^1(\boldsymbol{y}/H_1)+n_1A_j^1+O(|\boldsymbol{y}|)$, $|\boldsymbol{y}|\to 0$ $(j=1,2)$, we have $A_1^1=0$ and $A_2^1=d_0^1$, where the constant d_0^i is determined with the use of the results of [21, § 22] and has the form

$$d_0^i=\int_0^\infty \frac{1}{u}\Big(1-\mathrm{e}^{-u}-L_i(u)\Big)\,du,\qquad L_i(u)=\frac{2æ_i\sinh\,(2u)-4u}{2æ_i\cosh\,(2u)+1+æ_i^2+4u^2}.$$

In particular, $d_0^i\approx 0.527$ for Poisson's ratio $\nu_i=0.3$ (see [21, Table 3]).

The half-width of the contact zone is calculated from the formula

$$h_i^2=2R(n_i+n)P\qquad (i=1,2). \tag{5.9}$$

The contact force P is related to the approach of the punches by the formula

$$nP\Big(\ln\frac{4R}{h_1}+\ln\frac{4R}{h_2}-1\Big)+\sum_{i=1}^{2} n_iP\Big(\ln\frac{2H_i}{h_i}+\frac{1}{2}-d_0^i\Big)=2\delta_0. \tag{5.10}$$

The displacement of the disk center due to deformation is given by

$$\alpha = \frac{1}{4}\,nP\ln\frac{n_1+n}{n_2+n} + \sum_{i=1}^{2}\frac{(-1)^i}{2}\,n_iP\Big(\ln\frac{2H_i}{h_i} + \frac{1}{2} - d_0^i\Big). \tag{5.11}$$

Formulas (5.9)–(5.11) can be used to calculate elastic strains of rolling bearings.

6. Asymptotic Model of Quasistatic Collision of Plane Elastic Bodies. The quasistatic problem of collision of circular cylinders along their generatrix was solved in [22] (see also [23]). We construct an asymptotic model for the quasistatic collision of two cylinders along the common generatrix. Following [22, 23], we use the solution of the problem of the circular disks Ω^1 and Ω^2 (see Fig. 4) compressed by distributed loads with density $(-1)^{r+1}S_r^{-1}P\boldsymbol{e}_2$ $(r = 1, 2)$, where P is the contact-pressure resultant and $S_r = \pi R_r^2$ is the area of the disk Ω^r.

We write the external asymptotic representation in the form $\boldsymbol{v}^r(\boldsymbol{y}) = P\boldsymbol{G}^r(\boldsymbol{y}) + \alpha_r\boldsymbol{e}_2$. Here $\boldsymbol{G}^r(\boldsymbol{y})$ is a solution of the problem of deformation of the disk Ω^r under the distributed load $(-1)^{r+1}S_r^{-1}\boldsymbol{e}_2$ balanced by the unit force $(-1)^r\boldsymbol{e}_2$ applied at the point C. This solution satisfies the condition $\boldsymbol{G}^r(O_r) = 0$ at the point O_r (disk center). Using the explicit formulas given in [22], we obtain

$$\boldsymbol{G}^r(\boldsymbol{y}) = (-1)^{r+1}\Big[\boldsymbol{S}^r(\boldsymbol{y}/R_r) + n_rA_2^r\boldsymbol{e}_2\Big] + O(|\boldsymbol{y}|), \qquad |\boldsymbol{y}| \to 0; \tag{6.1}$$

$$A_2^r = \Big[2(æ_r + 1)\Big]^{-1}(æ_r + 2) \qquad (r = 1, 2). \tag{6.2}$$

Considering the boundary layer in a first approximation, we ignore the distributed loads. Then, the solution constructed in Sec. 4 yields the equation

$$\alpha = P\sum_{r=1}^{2} n_r\Big(\ln\frac{2R_r}{h} + \frac{1}{2} - A_2^r\Big), \tag{6.3}$$

which relates the approach of the disks $\alpha = \alpha_1 - \alpha_2$ to the contact force P. The half-width of the contact section h is calculated from formula (5.1).

Equation (6.3) [with allowance for (6.2)] coincides with a similar equation obtained by a different method in [22; 23, Chapter 3, Eq. (4.6)].

According to the second Newton law, the motion of the center of inertia of the disk Ω^r is governed by the equation $S_r\rho_r\ddot{\alpha}_r = (-1)^rP$, where ρ_r is the density of the material and the dot denotes differentiation with respect to time. A consequence of last two equations $(r = 1, 2)$ is the equation

$$M_0\ddot{\alpha} = -P \qquad [{}_0 = S_1S_2\rho_1\rho_2/(S_1\rho_1 + S_2\rho_2)]. \tag{6.4}$$

Equation (6.4) is supplemented by the initial conditions $\alpha = 0$ and $\dot{\alpha} = v_1 - v_2$, where v_r is the velocity of the body Ω^r at $t = 0$.

In [22, 23], Eqs. (5.1), (6.3), and (6.4) are used to construct quasistatic theory of colliding circular cylinders. To use the results of [22, 23] in the general case of central collision of plane bounded bodies which are in contact at a point at the initial moment, one need to calculate the coefficients in (6.2). In the case of an elastic ring, the quantity A_2^r can be determined from formulas of [24].

Conclusions. The coefficient A_2^2 in the initial problem can be expressed explicitly in the particular case of a circular disk by using the formulas of [12, § 80a].

The solutions constructed in Sec. 5 for circular disks are generalized to the case of elastic rings.

To refine the asymptotic solution obtained, it is necessary to construct an asymptotic representation for the boundary layer in the regions $\eta_2 \leqslant \sqrt{\varepsilon}\varphi_1(\eta_1)$ and $\eta_2 \geqslant \sqrt{\varepsilon}\varphi_2(\eta_1)$ for the problem of one-sided contact [see (4.3)]. Corrections to the vector $\boldsymbol{W}^r(\boldsymbol{\eta})$ should be determined with allowance for corrections to the contact segment $(-h_*, h_*)$. Generally, the variations of the contact zone are different in this case (as in the case of taking into account shear stresses in [25, 26]). It should be noted that formal asymptotic representations for one-sided contact zones was constructed in [27, 28].

The main result of this work is the following. The elastic strain of plane bodies Ω^1 and Ω^2 in contact, which cannot be expressed in terms of contact stresses in the Hertz theory (see [15, § 5.6]), is determined as a first approximation by calculating the dimensionless integral characteristics A_2^1 and A_2^2, which can be referred to as the local compliances of the elastic bodies Ω^1 and Ω^2, respectively.

This work was supported by the Administration of St. Petersburg (Grant No. M2000-2.2.P-16).

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